INVESTIGATION OF THE OUTPUT RESPONSE OF CLIPPERS TO GAUSSIAN INPUTS

STUART G. MURRAY

INVESTIGATION OF THE OUTPUT RESPONSE

OF

CLIPPERS TO GAUSSIAN INPUTS

* * * * *

Stuart G. Murray

This work is accepted as fulfilling

the thesis requirements for

the

CERTIFICATE OF COMPLETION

IN

ENGINEERING ELECTRONICS

from the

United States Naval Postgraduate School

PREFACE

The present writer had the honor and pleasure of being assigned to Hughes Tool Company, Culver City, California during the first three months of 1955. This assignment was his industrial experience tour from the United States Naval Postgraduate Shool, Monterey, California. While at Hughes, his work consisted of mathematically analizing a part of an electronic circuit.

Much is known about a non-linear circuit response to a signal; however, little is known about a non-linear circuit response to both a signal and a gaussian noise. The aim of this paper is to further the knowledge of the latter area in regard to clippers.

The writer wishes to thank Mr. Roderic C. Davis of Hughes Tool Company, Aircraft Division for his guidance, assistance, encouragement, and cooperation in the preparation of this paper.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS AND ABBREVIATIONS

$\sigma_1^{\ 2}$	Signal Energy of Channel One
$\sigma_2^{\ 2}$	Signal Energy of Channel Two
E	Mean Value or Mathematical Expectation Operator
$e_o(t)$	Output of Multiplier
α	Slope of Clipper Output vs Input Curve
D	Clipping Threshold
$\rho_s(\tau_o)$	Auto-correlation Function of s(t)
$\Phi^n(x)$	n^{th} Derivative of Gaussian Cumulative Distribution
S/N	Signal to Noise Power Ratio
$v_1(t)$	Input to Channel One
m(t)	Modulating Function
s(t)	Unmodulated Signal Function
$n_1(t)$	Noise Function in Channel One
$v_2(t)$	Input to Channel Two
τ_o	Time Between Signal Received by Channel One and Two
$m(t+\tau_o)$	Modulating Function Delayed by Time τ_o
$s(t+\tau_o)$	Unmodulated Signal Function Delayed by Time τ_o
$n_2(t)$	Noise Function in Channel Two
$e_1(t)$	Output of Channel One
$e_2(t)$	Output of Channel Two
$\sigma_s^{\ 2}$	Signal Energy
$\sigma_{n_1}^{\ 2}$	Noise Energy in Channel One
$\sigma_{n_2}^{\ 2}$	Noise Energy in Channel Two

ω	Frequency
$H_n(x)$	n^{th} Order Hermite Polynomial in x

CHAPTER I

INTRODUCTION

The response of an individual clipper to a sine wave signal plus gaussian noise has been analyzed by S. O. Rice [6]. However, little has been written about clippers acting together in an electronic circuit. L. Robin [7] has discussed clippers in response to a signal with noise, but under limited conditions. His discussion was based upon conditions in which the amplification factor of the device was fixed and the signal energies of the two channels σ_1^2 and σ_2^2 were equal.

This paper will be an extension of L. Robins' work [7], but will be based upon conditions with a variable amplification factor and with signal energies of the two channels σ_1^2 and σ_2^2 not equal. Figure 1 is a functional block diagram of the circuit that will be analyzed.

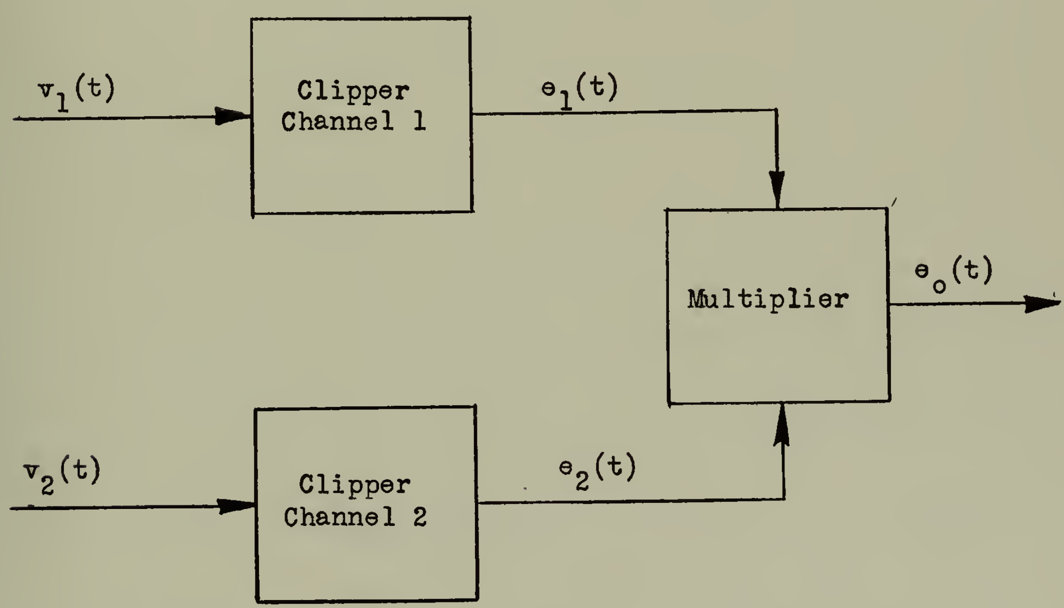

Figure 1. Functional Block Diagram

Two cases will be considered, both of which are for signals with continuous, stationary, gaussian noise inputs. The first case will consider ideal clippers. See Figure 2 for plot of response for ideal clippers. The second case will consider the actual clippers. See Figure 3 for plot of response for actual clippers.

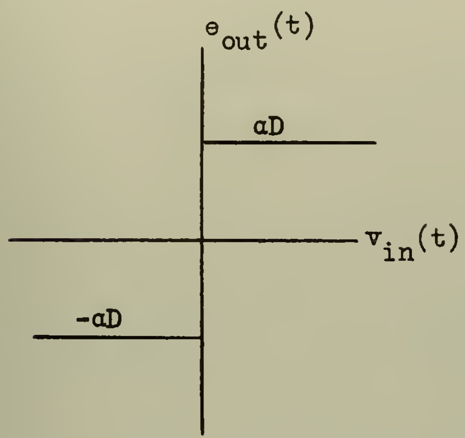

Figure 2. Ideal Clipper Response

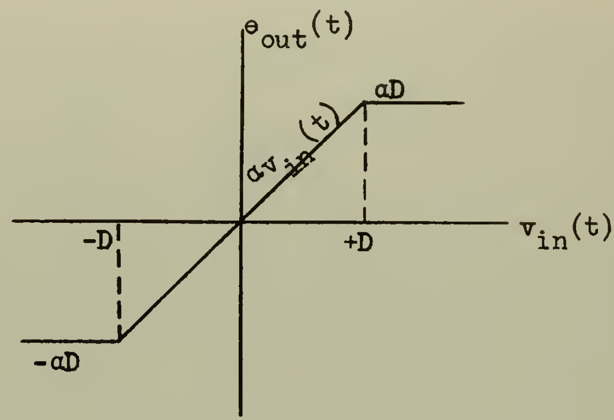

Figure 3. Actual Clipper Response

Chapter II considers the case of having ideal clippers in the two-channel system. Rice's method /6/ of solving the problem is applied to obtain a solution of

$$E\left[e_o(t)\right] = \frac{2\alpha^2 D^2}{\pi} \sin^{-1} \rho \qquad (2.16)$$

Chapter III considers the same case as Chapter II but a different statistical method is applied in order to obtain the solution. Notice that this second method is much shorter and easier than Rice's method in obtaining the solution that

$$E\left[e_o(t)\right] = \frac{2\alpha^2 D^2}{\pi} \sin^{-1} \rho \qquad (3.2)$$

Consideration of the actual clipper by a combination of the methods of Rice /6/ and Robin /7/ is discussed in Chapter IV. The solution can

be stated in integral form as

$$E\left[e_o(t)\right] = 2\rho a^2\left\{\int_0^D e^{-\frac{t^2}{2\sigma_1^2}}\,dt\right\}\left\{\int_0^D e^{-\frac{t^2}{2\sigma_2^2}}\,dt\right\}$$

$$+\frac{a^2\sigma_1\sigma_2}{\pi}\int_0^\rho \frac{\rho - t}{\sqrt{1 - t^2}}\, e^{-\frac{D^2}{2(1-t^2)}\left[\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right]}\sinh\frac{D^2 t}{\sigma_1\sigma_2(1-t^2)}\,dt \qquad (4.16)$$

or if the integration is performed and a series expansion is used, then

$$E\left[e_o(t)\right] = 4\rho a^2\sigma_1\sigma_2\left(\operatorname{erf}\frac{D}{\sigma_1\sqrt{2}}\right)\left(\operatorname{erf}\frac{D}{\sigma_2\sqrt{2}}\right)$$

$$+\frac{2a^2D^2}{\pi}\left\{\sin^{-1}\rho - \frac{D^2\rho\left[\sigma_1^2+\sigma_2^2\right]}{3\sigma_1^2\sigma_2^2(1-\rho^2)^{1/2}} + \frac{D^4(3\rho^3+8\rho)}{90(1-\rho^2)^{3/2}\sigma_1^2\sigma_2^2}\right.$$

$$\left.+\frac{D^4(\sigma_1^2+\sigma_2^2)(3\rho+5\rho^3-2\rho^5)}{120\,\sigma_1^4\,\sigma_2^4\,(1-\rho^2)^{5/2}} + \cdots\cdots\right\} \qquad (4.19)$$

where

$$\operatorname{erf}\frac{D}{\sigma_1\sqrt{2}} = \int_0^{D/\sigma_1\sqrt{2}} e^{-x^2}\,dx$$

and

$$\operatorname{erf}\frac{D}{\sigma_2\sqrt{2}} = \int_0^{D/\sigma_2\sqrt{2}} e^{-x^2}\,dx$$

Chapter V considers the actual clippers in the circuit by a different statistical method given by Cramér [2]. The same solution is obtained as was obtained in Chapter IV but is expressed in a different form. The present solution is

$$E\left[e_o(t)\right] = 4a^2\sigma_1\sigma_2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\,\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \qquad (5.31)$$

Of interest is the signal-to-noise power ratio of the input to this system compared to the output of the multiplier. The actual signal-to-noise ratio is

$$S/N = \frac{\left\{E\left[e_o(t)\right]\right\}^2}{E\left[e_o(t)\right]^2 - \left\{E\left[e_o(t)\right]\right\}^2} \qquad (6.2)$$

A solution for the S/N power ratio for both the actual and the ideal clippers was obtained. In the case of the actual clipper several attempts were made to reduce the expression, but these attempts were unsuccessful. However, in the case of the ideal clipper, the S/N ratio reduced to

$$S/N = \frac{\mu^2}{1 - \mu^2} \qquad \text{where} \qquad \mu = \frac{2}{\pi}\sin^{-1}\rho \qquad (3.7)$$

CHAPTER II

IDEAL CLIPPER METHOD I

The first case to be considered will be ideal clippers in the circuit of Figure 1. This analysis will use a method of mathematical analyzation similar to the one used in "Mathematical Analysis of Random Noise" by S. O. Rice [6]. Figure 4 and Figure 5 are the plots of the responses of the clippers in the two channels.

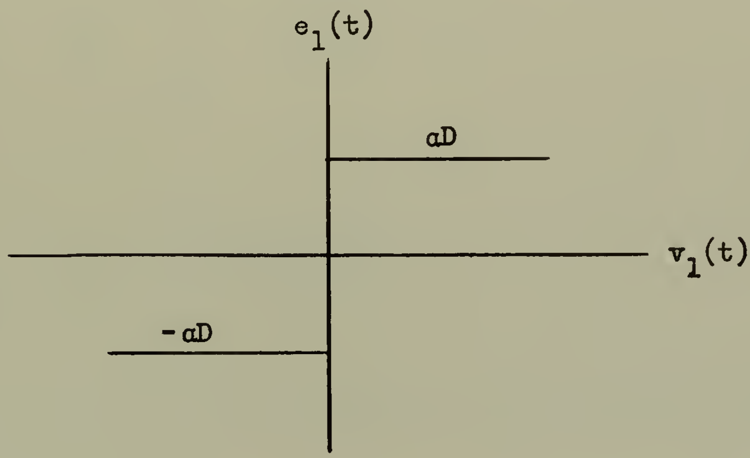

Figure 4. Plot of Ideal Clipper Response, Channel One

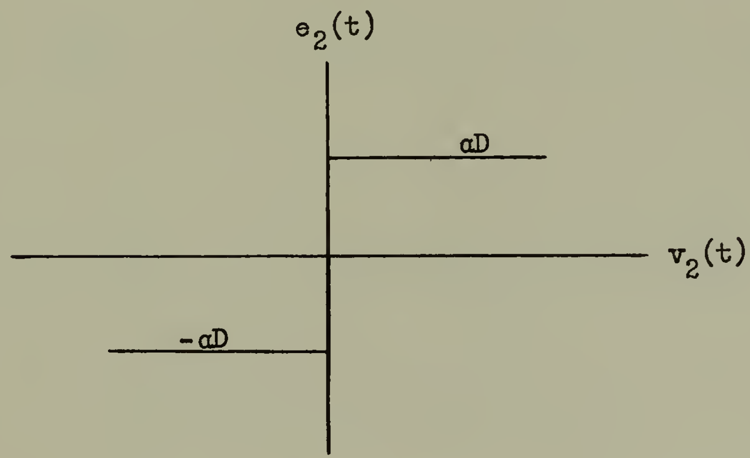

Figure 5. Plot of Ideal Clipper Response, Channel Two

$$\text{Let } v_1(t) = m(t)s(t) + n_1(t) \tag{2.1}$$

$$v_2(t) = m(t+\tau_o)\, s(t+\tau_o) + n_2(t) \tag{2.2}$$

where $s(t_1)$, $n_1(t_2)$, and $n_2(t_2)$ for all t_1 and t_2 are stationary gaussian noises which are mutually independent.

Also, the output of the multiplier is

$$e_o(t) = e_1(t)\, e_2(t) \tag{2.3}$$

but

$$e_1(t) = \frac{aD}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega_1 v_1(t)}}{\omega_1}\, d\omega_1 \tag{2.4}$$

and

$$e_2(t) = \frac{aD}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega_2 v_2(t)}}{\omega_2}\, d\omega_2 \tag{2.5}$$

Proof of the above two assumed equations is in Appendix I. Therefore, since $e_o(t) = e_1(t)\, e_2(t)$

$$e_o(t) = \left(\frac{aD}{\pi i}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\left[\omega_1 v_1(t) + \omega_2 v_2(t)\right]}}{\omega_1 \omega_2}\, d\omega_1\, d\omega_2 \tag{2.6}$$

then

$$e_o(t) = -\frac{a^2 D^2}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\left\{\omega_1\left[m(t)s(t)+n_1(t)\right] + \omega_2\left[m(t+\tau_o)s(t+\tau_o)+n_2(t)\right]\right\}}}{\omega_1 \omega_2}\, d\omega_1\, d\omega_2 \tag{2.7}$$

Since the modulating function m(t) is a non-random function, then

$$E\, e^{i\left[\omega_1 v_1(t)+\omega_2 v_2(t)\right]} = e^{-1/2 \sum_{i,j}^{2} \lambda_{ij}\omega_i\omega_j} \tag{2.8}$$

where $\lambda_{11} = E\,\overline{v_1(t)}^{2}$

$\lambda_{12} = \lambda_{21} = E\, v_1(t)v_2(t)$

$\lambda_{22} = E\,\overline{v_2(t)}^{2}$

This yields

$$E\, e^{i\left[\omega_1 v_1(t)+\omega_2 v_2(t)\right]} = e^{-1/2\left[\lambda_{11}\omega_1^2+2\lambda_{12}\omega_1\omega_2+\lambda_{22}\omega_2^2\right]} \tag{2.9}$$

therefore,

$$E\left[e_o(t)\right] = -\frac{\alpha^2 D^2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-1/2\left[\lambda_{11}\omega_1^2+2\lambda_{12}\omega_1\omega_2+\lambda_{22}\omega_2^2\right]}}{\omega_1\omega_2}\,d\omega_1 d\omega_2 \tag{2.10}$$

Note that a partial derivative of $E\left[e_o(t)\right]$ with respect to λ_{12} yields

$$\frac{\partial}{\partial\lambda_{12}} E\left[e_o(t)\right] = \frac{\alpha^2 D^2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-1/2\left[\lambda_{11}\omega_1^2+2\lambda_{12}\omega_1\omega_2+\lambda_{22}\omega_2^2\right]}\,d\omega_1 d\omega_2 \tag{2.11}$$

but remembering

$$\int_{-\infty}^{\infty}\cdots\cdots\int_{-\infty}^{\infty} e^{-Q(x_1,\ldots,x_n)/2}\,dx_1\ldots dx_m = \frac{(2\pi)^{n/2}}{\sqrt{A}} \tag{2.12}$$

where Q is a positive definite quadratic form, and the determinant of the associated matrix is A.

Then

$$\frac{\partial}{\partial\lambda_{12}} E\left[e_o(t)\right] = \frac{2a^2D^2}{\pi\sqrt{A}} \tag{2.13}$$

$$\text{where } A = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix}$$

and then $\sqrt{A} = \sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}$

therefore,

$$\frac{\partial}{\partial\lambda_{12}} E\left[e_o(t)\right] = \frac{2a^2D^2}{\pi\sqrt{\lambda_{11}\lambda_{22} - \lambda_{12}^2}} \tag{2.14}$$

Integrating Equation (2.14) with respect to λ_{12} gives

$$E\left[e_o(t)\right] = \frac{2a^2D^2}{\pi}\int_0^{\lambda_{12}} \frac{dx}{\sqrt{\lambda_{11}\lambda_{22} - x^2}}$$

$$= \frac{2a^2D^2}{\pi} \sin^{-1}\left(\frac{\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}}\right) \tag{2.15}$$

but

$$\lambda_{11} = E\,\overline{v_1(t)}^2 = \left\{\left[m(t)\right]^2 \sigma_s^2 + \sigma_{n_1}^2\right\}$$

$$\lambda_{12} = \lambda_{21} = Ev_1(t)v_2(t) = m(t)m(t+\tau_o)\sigma_s^2\rho_s(\tau_o)$$

and

$$\lambda_{22} = E\,\overline{v_2(t)}^2 = \left\{\left[m(t+\tau_o)\right]^2 \sigma_s^2 + \sigma_{n_1}^2\right\}$$

This can be further reduced by letting

$$\frac{\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}} = \rho$$

and then substituting back into Equation (2.15), which yields

$$E\left[\theta_o(t)\right] = \frac{2\alpha^2 D^2}{\pi} \sin^{-1} \rho \tag{2.16}$$

IDEAL CLIPPER METHOD II

The same problem that was solved in Chapter II by Rice's method [6] will be solved in this Chapter by Cramèr's method [2]. Hence, by applying Cramèr's method directly, it will be possible to make a plot of the mass distribution of the frequency function $f(v_1, v_2)$. See Figure 6.

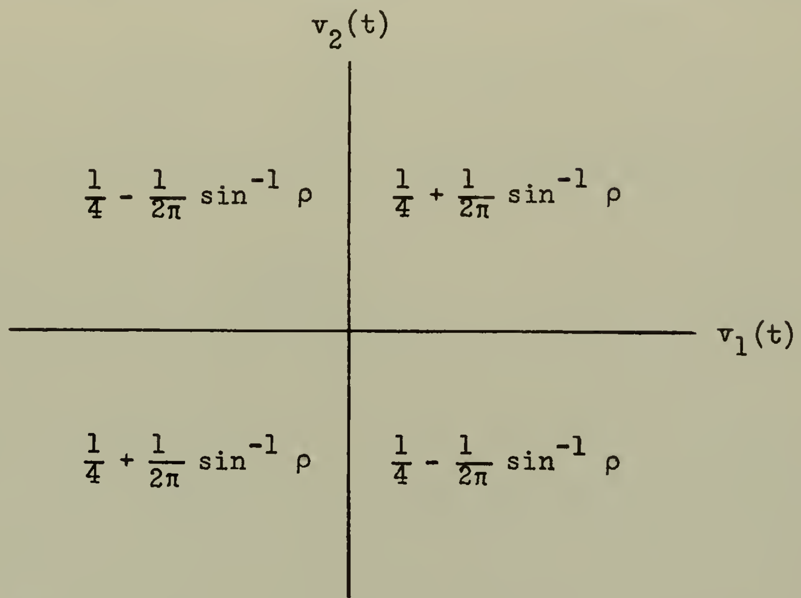

Figure 6. Mass Distribution of Frequency Function $f(v_1, v_2)$

Notice that the contribution to $Ee_1(t)e_2(t)$ from the first and third quadrants will be $2\alpha^2D^2\left[\frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho\right]$. From the second and fourth quadrants, one obtains $-2\alpha^2D^2\left[\frac{1}{4} - \frac{1}{2\pi} \sin^{-1} \rho\right]$. Adding these two results together gives the total contribution as

$$E\, e_1(t)e_2(t) = 2\alpha^2D^2\left[\frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho\right] - 2\alpha^2D^2\left[\frac{1}{4} - \frac{1}{2\pi} \sin^{-1} \rho\right] \quad (3.1)$$

therefore, since $e_o(t) = e_1(t)e_2(t)$

$$E\left[e_o(t)\right] = \frac{2\alpha^2D^2}{\pi} \sin^{-1} \rho \quad (3.2)$$

The first moment, $E\left[e_o(t)\right]$, was found in the above portion of this chapter. Now the second moment, $E\left[\overline{e_o(t)}^2\right]$, will be obtained. See Figure 7 for plots of the probability distribution of the ideal clipper's functions. Parts A and B of Figure 7 are plots of $e_i(t)$ versus $v_1(t)$ and $e_2(t)$ versus $v_2(t)$, while parts C and D are plots of $\overline{e_1(t)}^2$ versus $v_1(t)$ and $\overline{e_2(t)}^2$ versus $v_2(t)$. The plot of $\overline{e_1(t)}^2$ versus $\overline{e_2(t)}^2$ is Figure 7, part E. To obtain the second moment, $E\left[\overline{e_o(t)}^2\right]$by Cramèr's method, notice that $E\left[\overline{e_o(t)}^2\right] = E\left[\overline{e_1(t)}^2\ \overline{e_2(t)}^2\right]$. The sum of the four individual quadrants of Figure 7, part E will yield $E\left[\overline{e_1(t)}^2\ \overline{e_2(t)}^2\right]$.

Now

$$E\left[\overline{e_1(t)}^2\ \overline{e_2(t)}^2\right] = 4\left[\frac{\alpha^4 D^4}{4}\right] \tag{3.3}$$

Therefore,

$$E\left[\overline{e_o(t)}^2\right] = \alpha^4 D^4 \tag{3.4}$$

At this point it is of interest to compute the signal-to-noise ratio. The S/N ratio for the ideal clipper is:

$$S/N = \frac{\left\{E\left[e_o(t)\right]\right\}^2}{E\left[\overline{e_o(t)}^2\right] - \left\{E\left[e_o(t)\right]\right\}^2} \tag{3.5}$$

Now substituting in the above equation the values found in equations (3.2) and (3.4), one gets for the ideal clippers

$$S/N = \frac{\left[\frac{2\alpha^2 D^2}{\pi} \sin^{-1} \rho\right]^2}{\alpha^4 D^4 - \left[\frac{2\alpha^2 D^2}{\pi} \sin^{-1} \rho\right]^2} \tag{3.6}$$

$$= \frac{\left[\frac{2}{\pi} \sin^{-1} \rho\right]^2}{1 - \left[\frac{2}{\pi} \sin^{-1} \rho\right]^2}$$

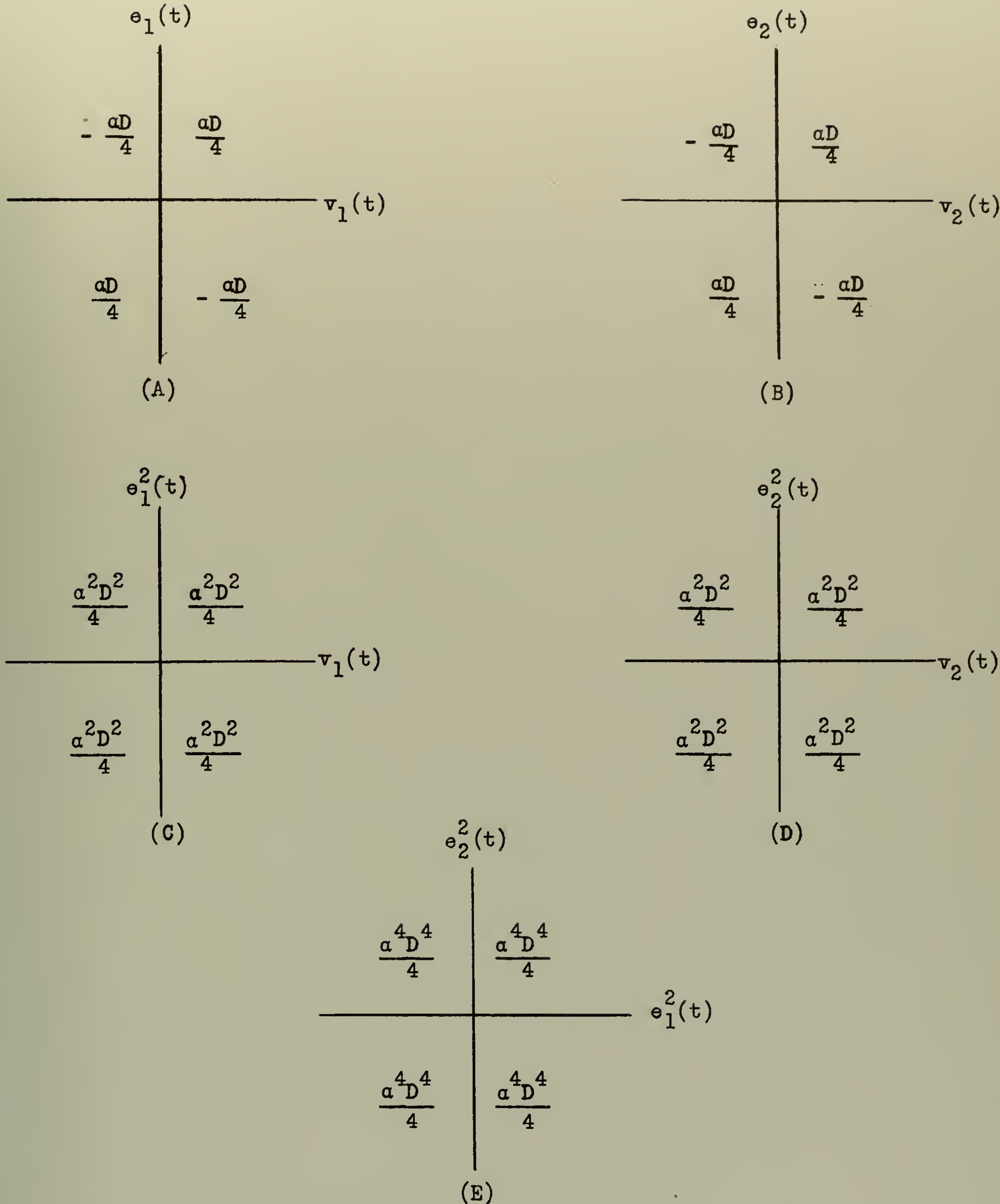

Figure 7. Probability Distribution of Ideal Clipper Functions

If we let $\mu = \frac{2}{\pi} \sin^{-1} \rho$, then for the ideal clippers

$$S/N = \frac{\mu^2}{1 - \mu^2} \tag{3.7}$$

CHAPTER IV

ACTUAL CLIPPER METHOD I

In this chapter will be analyzed the circuit of Figure 1 where the responses of the actual clippers are represented in Figures 8 and 9. The approach to the problem will follow Rice's method [6] until equation (4.12) is reached. At this point Robins' method [7] will be used to complete the solution.

Now consider the inputs,

$$v_1(t) = m(t)\ s(t) + n_1(t)$$

$$v_2(t) = m(t+\tau_o)\ s(t+\tau_o) + n_2(t)$$

where $s(t_1)$, $n_1(t_2)$ and $n_2(t_2)$ for all t_1 and t_2 are stationary gaussian noises which are mutually independent.

But $e_o(t) = e_1(t)\ e_2(t)$

and $e_1(t)$ can be written as

$$e_1(t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\sin v_1(t)\ \mu \sin D\mu}{\mu^2}\, d\mu \qquad (4.1)$$

also

$$e_2(t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\sin v_2(t)\ x \sin Dx}{x^2}\, dx \qquad (4.2)$$

Proof of the above two assumed equations is in Appendix II. Combining and letting $\mu = \omega_1$ and $x = \omega_2$, then

$$e_o(t) = e_1(t)\ e_2(t)$$

$$= \frac{a^2}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin D\omega_1 \sin D\omega_2 \sin v_1(t)\omega_1 \sin v_2(t)\omega_2}{\omega_1^2\ \omega_2^2}\, d\omega_1 d\omega_2 \qquad (4.3)$$

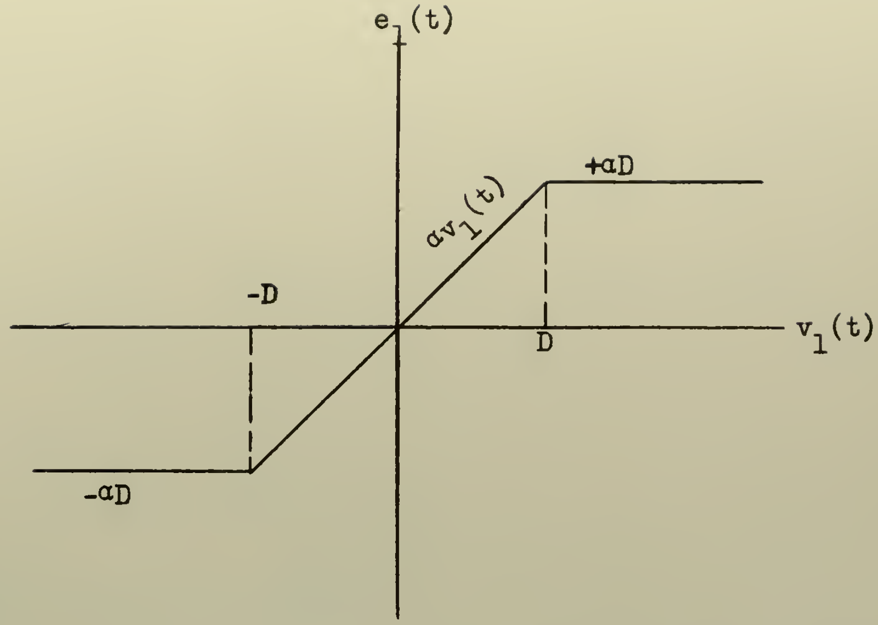

Figure 8. Plot of Actual Clipper Response, Channel One

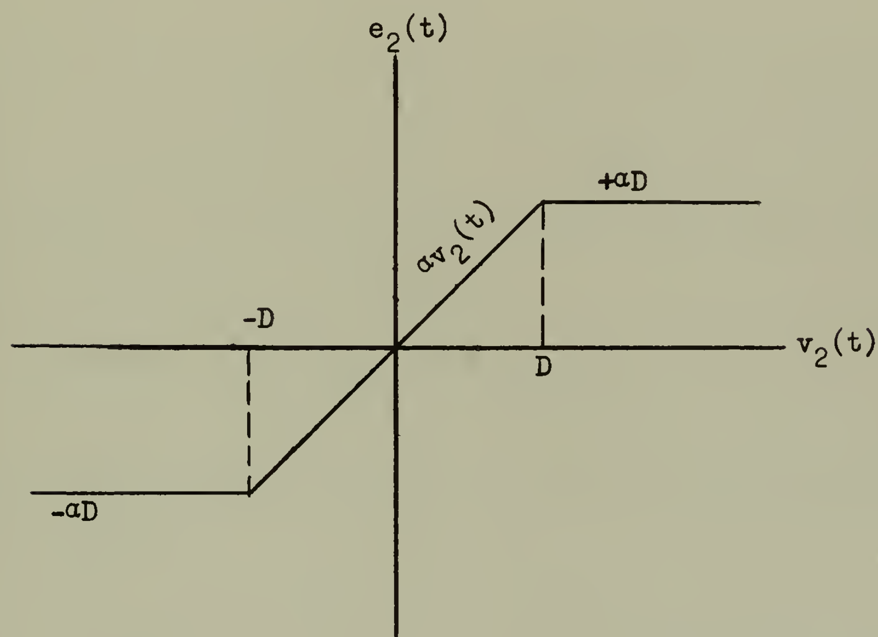

Figure 9. Plot of Actual Clipper Response, Channel Two

Taking the expected value of each side of the above equation will get

$$E\left[e_o(t)\right] = \frac{a^2}{\pi^2} E \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin D\omega_1 \ \sin D\omega_2 \ \sin v_1(t)\omega_1 \ \sin v_2(t)\omega_2}{\omega_1^2 \ \omega_2^2} d\omega_1 \ d\omega_2 \tag{4.4}$$

$$E\left[e_o(t)\right] = \frac{-a^2}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin D\omega_1 \ \sin D\omega_2}{\omega_1^2 \ \omega_2^2} \Big\{ E\, e^{i(v_1\omega_1 + v_2\omega_2)} + E e^{-i(v_1\omega_1 + v_2\omega_2)} - E e^{-i(v_1\omega_1 - \omega_2 v_2)} - E e^{i(v_1\omega_1 - v_2\omega_2)} \Big\} d\omega_1 d\omega_2 \tag{4.5}$$

Using equation (2.8), one obtains

$$E\left[e_o(t)\right] = -\frac{2a^2}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin D\omega_1 \ \sin D\omega_2}{\omega_1^2 \ \omega_2^2} \Big\{ E e^{i(v_1\omega_1 + v_2\omega_2)} - E e^{i(v_1\omega_1 - v_2\omega_2)} \Big\} d\omega_1 \ d\omega_2 \tag{4.6}$$

Now since both $v_1(t)$ and $v_2(t)$ possess a joint gaussian probability distribution, it is possible to write the following from equation (2.9):

$$E\left[e_o(t)\right] = \frac{-2a^2}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin D\omega_1 \ \sin D\omega_2}{\omega_1^2 \ \omega_2^2} \Big\{ e^{-\frac{1}{2}(\lambda_{11}\omega_1^2 + 2\lambda_{12}\omega_1\omega_2 + \lambda_{22}\omega_2^2)} - e^{-\frac{1}{2}(\lambda_{11}\omega_1^2 - 2\lambda_{12}\omega_1\omega_2 + \lambda_{22}\omega_2^2)} \Big\} d\omega_1 \ d\omega_2 \tag{4.7}$$

Note that the second partial derivative of the above equation taken with respect to λ_{12} gives

$$\frac{\partial^2}{\partial\lambda_{12}^2} E\left[e_o(t)\right] = \frac{-2a^2}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin D\omega_1 \ \sin D\omega_2 \Big\{ e^{-\frac{1}{2}(\lambda_{11}\omega_1^2 + 2\lambda_{12}\omega_1\omega_2 + \lambda_{22}\omega_2^2)} - e^{-\frac{1}{2}(\lambda_{11}\omega_1^2 \mp 2\lambda_{12}\omega_1\omega_2 + \lambda_{22}\omega_2^2)} \Big\} d\omega_1 \ d\omega_2 \tag{4.8}$$

To reduce to standard notation, however, let

$$\lambda_{11} = \sigma_1^2, \quad \lambda_{12} = \rho\sigma_1\sigma_2, \text{ and } \lambda_{22} = \sigma_2^2$$

so that

$$Ee^{i(v_1\omega_1+v_2\omega_2)} = e^{\frac{1}{2}(\sigma_1^2\omega_1^2 + 2\rho\sigma_1\sigma_2\,\omega_1\omega_2+\sigma_2^2\omega_2^2)} \tag{4.9}$$

hence from equation (2.12)

$$\frac{\partial^2}{\partial\rho^2} E\left[e_o(t)\right] = \frac{-2\alpha^2\sigma_1^2\sigma_2^2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin D\omega_1 \sin D\omega_2\, e^{\frac{1}{2}Q(\omega_1\omega_2)}\, d\omega_1\, d\omega_2 \tag{4.10}$$

Now consider the $\sin D\omega_1 \sin D\omega_2$ term by itself:

$$4 \sin D\omega_1 \sin D\omega_2 = -(e^{iD\omega_1} - e^{-iD\omega_1})(e^{iD\omega_2} - e^{-iD\omega_2})$$

$$= -e^{iD(\omega_1+\omega_2)} - e^{-iD(\omega_1+\omega_2)} + e^{iD(\omega_1-\omega_2)} - e^{-iD(\omega_1-\omega_2)} \tag{4.11}$$

Noting equation (4.11), it can be seen that since $Q^{-1}(t_1t_2)$ is a symmetric quadratic in t_1 and t_2, the contributions from the first and second terms are equal, and similarly, the contributions from the third and fourth terms are equal.

From Cramèr [2] it is noted that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i\sum_1^n t_jx_j - \frac{1}{2}Q(x_1\ldots x_n)}\, dx_1\ldots dx_n = \frac{(2\pi)^{n/2}}{\sqrt{A}}\, e^{-\frac{1}{2}Q^{-1}(t_1\ldots t_n)}$$

where Q is a positive definite quadratic form, A is the determinant of the associated matrix, and Q^{-1} is the reciprocal form to Q. In this case,

$$Q(\omega_1,\omega_2) = \sigma_1^2\omega_1^2 + 2\rho\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2$$

therefore,

$$Q^{-1}(t_1,t_2) = \frac{1}{1-\rho^2}\left[\frac{t_1^2}{\sigma_1^2} - \frac{2\rho t_1 t_2}{\sigma_1\sigma_2} + \frac{t_2^2}{\sigma_2^2}\right]$$

and

$$A = \begin{vmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2\,(1-\rho^2)$$

$$\sqrt{A} = \sigma_1\sigma_2\sqrt{1-\rho^2}$$

Combining, but remembering that $t_1 = t_2 = D$ in our case to obtain

$$\frac{\partial^2}{\partial\rho^2} E\left[e_o(t)\right] = \frac{\alpha^2\ \sigma_1\sigma_2}{\pi\sqrt{1-\rho^2}}\left\{e^{\frac{-D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2} - \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right]} - e^{\frac{-D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2} + \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right]}\right\} \tag{4.12}$$

In order to obtain $E\left[e_o(t)\right]$ it is necessary to take a double integral of both sides of equation (4.12). But notice that equation (4.12) is in the form of

$$\frac{\partial^2}{\partial x^2}\ \psi(x) = F(x)$$

but

$$\int_0^\rho \frac{\partial^2}{\partial x^2}\ \psi(x)\ dx = \frac{\partial}{\partial\rho}\ \psi(\rho) - \frac{\partial}{\partial x}\ \psi(x)\Big|_{x=0}$$

$$= \int_0^\rho F(x)\ dx$$

Then $$\frac{\partial}{\partial\rho}\ \psi(\rho) = \frac{\partial}{\partial x}\ \psi(x)\Big|_{x=0} + \int_0^\rho F(x)\ dx$$

and

$$\int_0^\rho \frac{\partial}{\partial x}\ \psi(x)\ dx = \psi(\rho) - \psi(0)$$

$$= \rho\ \frac{\partial}{\partial x}\psi(x)\Big|_{x=0} + \int_0^\rho ds \int_0^s F(x)\ dx$$

Since for this case

$$\psi(x)\Big|_{x=0} = 0$$

and

$$\frac{\partial}{\partial x}\psi(x)\Big|_{x=0} = \frac{2\pi}{\sigma_1\sigma_2}\left\{\int_0^D e^{-\frac{t^2}{2\sigma_1^2}}\,dt\right\}\left\{\int_0^D e^{-\frac{t^2}{2\sigma_2^2}}\,dt\right\} \tag{4.13}$$

$$= 4\pi\left(\operatorname{erf}\frac{D}{\sigma_1\sqrt{2}}\right)\left(\operatorname{erf}\frac{D}{\sigma_2\sqrt{2}}\right) \tag{4.14}$$

The proof of equations (4.13) and (4.14) is in Appendix V, therefore,

$$\psi(x) = 4\pi\rho\left(\operatorname{erf}\frac{D}{\sigma_1\sqrt{2}}\right)\left(\operatorname{erf}\frac{D}{\sigma_2\sqrt{2}}\right) + \int_0^\rho ds\int_0^s F(x)\,dx \tag{4.15}$$

Now applying to equation (4.12), one obtains:

$$E\left[e_o(t)\right] = 2a^2\rho\left\{\int_0^D e^{-\frac{t^2}{2\sigma_1^2}}\,dt\right\}\left\{\int_0^D e^{-\frac{t^2}{2\sigma_2^2}}\,dt\right\} +$$

$$\frac{a^2\sigma_1\sigma_2}{\pi}\int_0^\rho ds\int_0^s \frac{dt}{\sqrt{1-t^2}}\left\{e^{-\frac{D^2}{2(1-t^2)}\left[\frac{1}{\sigma_1^2}+\frac{-2t}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]} -\right.$$

$$\left. e^{\frac{-D^2}{2(1-t^2)}\left[\frac{1}{\sigma_1^2}+\frac{2t}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]}\right. \tag{4.16}$$

But if the order of integration is changed, the preceding equation reduces to:

$$E\left[e_o(t)\right] = \rho 4a^2\sigma_1\sigma_2\left(\operatorname{erf}\frac{D}{\sigma_1\sqrt{2}}\right)\left(\operatorname{erf}\frac{D}{\sigma_2\sqrt{2}}\right) +$$

$$\frac{a^2\sigma_1\sigma_2}{\pi}\int_0^\rho \frac{\rho - t}{\sqrt{1-t^2}}\left[e^{-\frac{D^2}{2(1-t^2)}\left(\frac{1}{\sigma_1^2}-\frac{2t}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right)} -\right.$$

$$\left. e^{-\frac{-D^2}{2(1-t^2)}\left(\frac{1}{\sigma_1^2}+\frac{2t}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right)}\right]dt \tag{4.17}$$

Consider the exponential term only. It is of the form $e^{-x} - e^{-y}$.

Expand this in a series, then

$$e^{-x} - e^{-y} = (y - x) + \frac{x^2 - y^2}{2!} + \frac{y^3 - x^3}{3!} + \frac{x^4 - y^4}{4!} + \cdots$$

but

$$x = \frac{D^2}{2(1 - t^2)} \left\{ \frac{1}{\sigma_1^2} - \frac{2t}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2} \right\}$$

and

$$y = \frac{D^2}{2(1 - t)^2} \left\{ \frac{1}{\sigma_1^2} + \frac{2t}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2} \right\}$$

then:

$$y - x = \frac{2D^2}{\sigma_1 \sigma_2} \left[\frac{t}{1 - t^2} \right]$$

$$x^2 - y^2 = \frac{-2D^4}{\sigma_1^3 \sigma_2^3} \left[\sigma_1^2 + \sigma_2^2 \right] \left[\frac{t}{(1 - t^2)^2} \right]$$

and

$$y^3 - x^3 = \frac{D^6}{12\sigma_1 \sigma_2} \left\{ \frac{4}{\sigma_1^2 \sigma_2^2} \left[\frac{t^3}{(1 - t^2)^3} \right] + \frac{3}{\sigma_1^4 \sigma_2^4} \left[\sigma_1^2 + \sigma_2^2 \right]^2 \left[\frac{t}{(1 - t^2)^3} \right] \right\}$$

Thus the first three terms of the expansion will be

$$e^{-\frac{D^2}{2(1-t^2)} \left\{ \frac{1}{\sigma_1^2} - \frac{2t}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2} \right\}} - e^{-\frac{D^2}{2(1-t^2)} \left\{ \frac{1}{\sigma_1^2} + \frac{2t}{\sigma_1 \sigma_2} + \frac{1}{\sigma_2^2} \right\}} =$$

$$\left\{ \frac{2D^2}{\sigma_1 \sigma_2} \left[\frac{t}{1 - t^2} \right] - \frac{2D^4}{\sigma_1^3 \sigma_2^3} \left[\sigma_1^2 + \sigma_2^2 \right] \left[\frac{t}{(1 - t^2)^2} \right] + \right.$$

$$\left. \frac{D^6}{12\sigma_1 \sigma_2} \left(\frac{4}{\sigma_1^2 \sigma_2^2} \left[\frac{t^3}{(1 - t^2)^3} \right] + \frac{3}{\sigma_1^4 \sigma_2^4} \left[\sigma_1^2 + \sigma_2^2 \right]^2 \left[\frac{t}{(1 - t^2)^3} \right] \right) + \cdots \right\}$$

which equals

$$\frac{2D^2}{\sigma_1\sigma_2}\left\{\frac{t}{1 - t^2} - \frac{D^2}{\sigma_1^2\sigma_2^2}\left[\sigma_1^2 + \sigma_2^2\right]\left[\frac{t}{(1 - t^2)^2}\right] + \frac{D^4}{24}\left(\frac{4}{\sigma_1^2\sigma_2^2}\left[\frac{t^3}{(1 - t^2)^3}\right] + \right.\right.$$

$$\left.\left.\frac{3}{\sigma_1^4\sigma_2^4}\left[\sigma_1^2 + \sigma_2^2\right]^2\left[\frac{t}{(1 - t^2)^3}\right]\right) + \ldots\ldots\right\}$$

Substituting back into equation (4.17) we then have

$$E\left[e_o(t)\right] = 4\rho a^2\sigma_1\sigma_2 \left(\operatorname{erf} \frac{D}{\sigma_2\sqrt{2}}\right)\left(\operatorname{erf} \frac{D}{\sigma_1\sqrt{2}}\right) +$$

$$\frac{2D^2a^2}{\pi}\int_0^\rho \frac{\rho - t}{(1 - t^2)^{1/2}}\left\{\frac{t}{1 - t^2} - \frac{D^2}{\sigma_1^2\sigma_2^2}\left[\sigma_1^2 + \sigma_2^2\right]\left[\frac{t}{(1 - t^2)^2}\right] + \right.$$

$$\left.\frac{D^4}{24}\left(\frac{4}{\sigma_1^2\sigma_2^2}\left[\frac{t^3}{(1 - t^2)^3}\right] + \frac{3}{\sigma_1^4\sigma_2^4}\left[\sigma_1^2 + \sigma_2^2\right]^2\left[\frac{t}{(1 - t^2)^3}\right]\right) + \ldots\right\} dt \tag{4.18}$$

The first term is known as the erf or error functions, and is tabulated. The second term will be integrated term by term, thus:

$$E\left[e_o(t)\right] = 4\rho a^2\sigma_1\sigma_2 \left(\operatorname{erf} \frac{D}{\sigma_1\sqrt{2}}\right)\left(\operatorname{erf} \frac{D}{\sigma_2\sqrt{2}}\right) +$$

$$\frac{2a^2D^2}{\pi}\left\{\sin^{-1}\rho - \frac{D^2}{\sigma_1^2\sigma_2^2}\left[\sigma_1^2 + \sigma_2^2\right]\frac{\rho}{3(1 - \rho^2)^{1/2}} + \right.$$

$$\frac{D^4}{24}\left[\frac{4\rho(3\rho + 8)}{\sigma_1^2\sigma_2^2\, 15(1 - \rho^2)^{3/2}}\right] +$$

$$\left.\frac{3}{\sigma_1^4\sigma_2^4}(\sigma_1^2 + \sigma_2^2)^2\,\frac{3\rho + 5\rho^3 - 2\rho^5}{15\,(1 - \rho^2)^{5/2}} + \ldots\right\} \tag{4.19}$$

CHAPTER V

ACTUAL CLIPPER METHOD II

The actual clipper in the circuit will again be considered in this chapter. Cramèr's method of analysis [2] will be used. It will be noted that the Φ functions are easier to handle than the integrals of the preceding chapter. In using this method, it is convenient to divide the plane representing the inputs $v_1(t)$ and $v_2(t)$ into sub-regions, and compute the contribution within each sub-region to the various moments of the output $e_o(t)$ of the multiplier. See Figure 10 for a plot of those sub-regions. Later, these contributions will be added in order to obtain the final solution of the moments of $e_o(t)$.

Assume $f(v_1, v_2)$ is of the form,

$$f(v_1, v_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{v_1^2}{\sigma_1^2} - \frac{2\rho v_1 v_2}{\sigma_1\sigma_2} + \frac{v_2^2}{\sigma_2^2}\right]\right\} \qquad (5.1)$$

but this is equal to:

$$= \frac{1}{\sigma_1\sigma_2} \sum_{\eta=0}^{\infty} \frac{\Phi^{(\eta+1)}\left(\frac{v_1}{\sigma_1}\right) \Phi^{(\eta+1)}\left(\frac{v_2}{\sigma_2}\right)}{\cancel{\eta^2}\ \eta!} \rho^{\eta} \qquad (5.2)$$

$$\text{where} \quad \Phi\left(\frac{x}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma}} e^{-\frac{t^2}{2}}\, dt$$

Consider the areas that are numbered in Figure 10, and write the equation of $E\left[e_o(t)\right]$ for each area. Then the sum of these individual contributions will represent the total area.

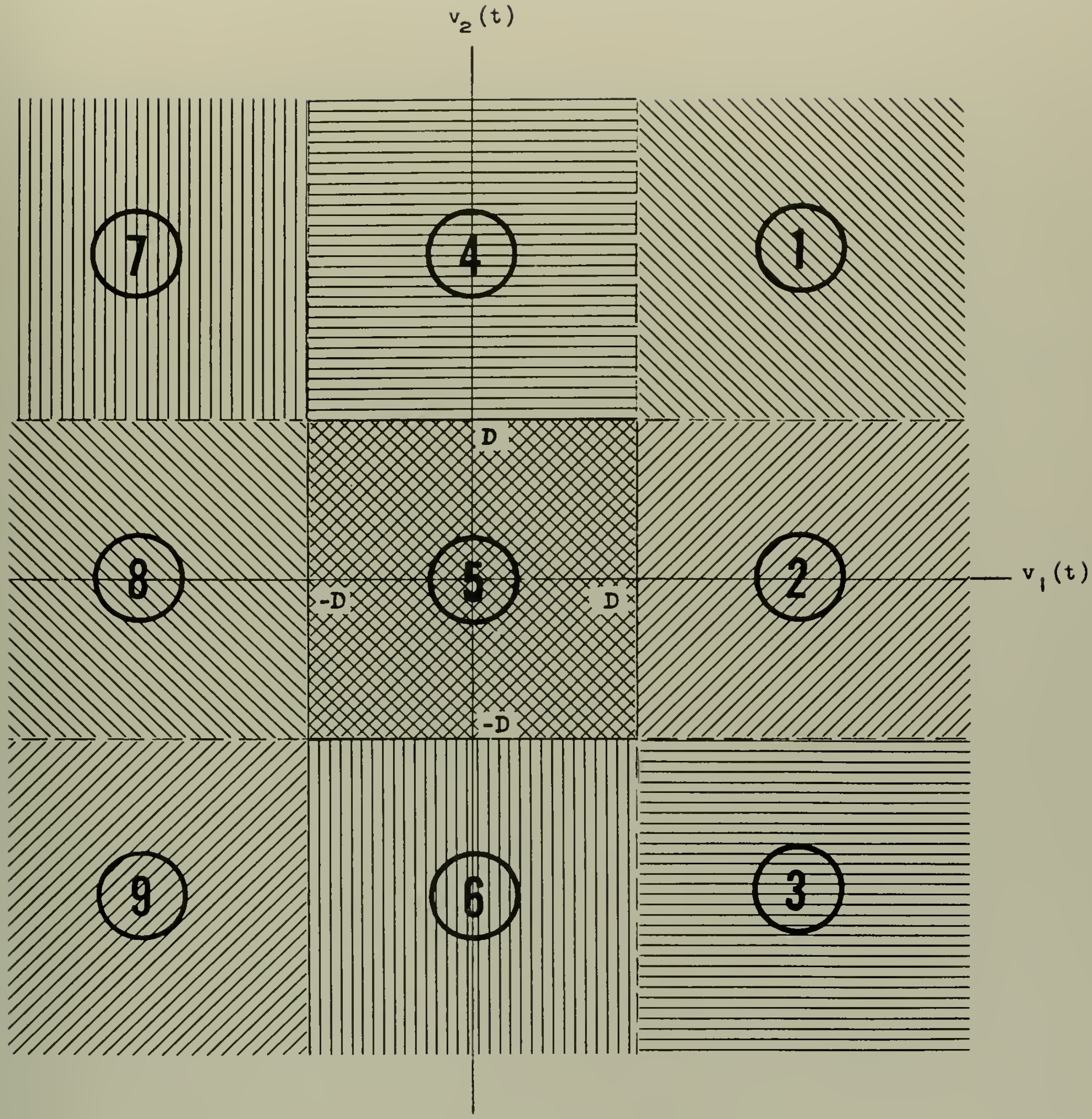

Figure 10. Plot of Sub-Regions of Inputs

Area 1:

$$E\left[e_{o_1}(t)\right] = \alpha^2 D^2 \int_D^\infty \int_D^\infty f(v_1, v_2)dv_1\, dv_2 \tag{5.3}$$

Area 2:

$$E\left[e_{o_2}(t)\right] = \alpha^2 D \int_D^\infty dv_1 \int_{-D}^{D} v_2\, f(v_1, v_2)dv_2 \tag{5.4}$$

Area 3:

$$E\left[e_{o_3}(t)\right] = -\alpha^2 D^2 \int_{-\infty}^{-D} dv_2 \int_D^\infty f(v_1, v_2)dv_1 \tag{5.5}$$

Area 4:

$$E\left[e_{o_4}(t)\right] = \alpha^2 D \int_{-D}^{D} dv_1 \int_D^\infty v_1\, f(v_1, v_2)\, dv_2 \tag{5.6}$$

Area 5:

$$E\left[e_{o_5}(t)\right] = \alpha^2 \int_{-D}^{D} \int_{-D}^{D} v_1 v_2\, f(v_1, v_2)\, dv_1\, dv_2 \tag{5.7}$$

Area 6:

$$E\left[e_{o_6}(t)\right] = -\alpha^2 D \int_{-D}^{D} dv_1 \int_{-\infty}^{-D} v_1\, f(v_1, v_2)\, dv_2 \tag{5.8}$$

Area 7:

$$E\left[e_{o_7}(t)\right] = -\alpha^2 D^2 \int_{-\infty}^{-D} dv_1 \int_D^\infty f(v_1, v_2)\, dv_2 \tag{5.9}$$

Area 8:

$$E\left[e_{o_8}(t)\right] = -\alpha^2 D \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2\, f(v_1, v_2)\, dv_2 \qquad (5.10)$$

Area 9:

$$E\left[e_{o_9}(t)\right] = \alpha^2 D^2 \int_{-\infty}^{-D} \int_{-\infty}^{-D} f(v_1, v_2)\, dv_1 dv_2 \qquad (5.11)$$

but

$$E\left[e_o(t)\right] = E\left[e_{o_1}(t)\right] + E\left[e_{o_2}(t)\right] + E\left[e_{o_3}(t)\right] + E\left[e_{o_4}(t)\right] + E\left[e_{o_5}(t)\right] + E\left[e_{o_6}(t)\right] + E\left[e_{o_7}(t)\right] + E\left[e_{o_8}(t)\right] + E\left[e_{o_9}(t)\right]$$

then

$$E\left[e_o(t)\right] = \alpha^2 \int_{-D}^{D}\int_{-D}^{D} v_1 v_2\, f(v_1, v_2)\, dv_1 dv_2 + \alpha^2 D^2 \int_{D}^{\infty}\int_{D}^{\infty} f(v_1, v_2) dv_1 dv_2$$

$$+ \alpha^2 D^2 \int_{-\infty}^{-D}\int_{-\infty}^{-D} f(v_1, v_2) dv_1 dv_2 - \alpha^2 D^2 \int_{D}^{\infty} dv_1 \int_{-\infty}^{-D} f(v_1, v_2) dv_2$$

$$-\alpha^2 D^2 \int_{-\infty}^{-D} dv_1 \int_{D}^{\infty} f(v_1, v_2) dv_2 + \alpha^2 D \int_{D}^{\infty} dv_1 \int_{-D}^{D} v_2\, f(v_1, v_2) dv_2$$

$$-\alpha^2 D \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2\, f(v_1, v_2) dv_2 + \alpha^2 D \int_{D}^{\infty} dv_2 \int_{-D}^{D} v_1\, f(v_1, v_2) dv_1$$

$$- \alpha^2 D \int_{-\infty}^{-D} dv_2 \int_{-D}^{D} v_1 \, f(v_1, v_2) dv_1 \qquad (5.12)$$

Now the above equation will be considered term by term.

The following terms are yielded from the basic assumed Equation (5.2),

$$\alpha^2 D^2 \int_D^{\infty}\int_D^{\infty} f(v_1, v_2) dv_1 dv_2 = \alpha^2 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{D}{\sigma_1}\right) \Phi^\eta\left(\frac{D}{\sigma_2}\right) \qquad (5.13)$$

$$\alpha^2 D^2 \int_{-\infty}^{-D}\int_{-\infty}^{-D} f(v_1, v_2) dv_1 dv_2 = \alpha^2 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{-D}{\sigma_1}\right) \Phi^\eta\left(\frac{-D}{\sigma_2}\right) \qquad (5.14)$$

$$-\alpha^2 D^2 \int_D^{\infty} dv_1 \int_{-\infty}^{-D} f(v_1, v_2) dv_2 = \alpha^2 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{D}{\sigma_1}\right) \Phi^\eta\left(\frac{-D}{\sigma_2}\right) \qquad (5.15)$$

$$-\alpha^2 D^2 \int_{-\infty}^{-D} dv_1 \int_D^{\infty} f(v_1, v_2) dv_2 = \alpha^2 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{-D}{\sigma_1}\right) \Phi^\eta\left(\frac{D}{\sigma_2}\right) \qquad (5.16)$$

but observe that these can be combined to give

$$\alpha^2 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) + \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right]\left[\Phi^\eta\left(\frac{D}{\sigma_2}\right) + \Phi^\eta\left(\frac{-D}{\sigma_2}\right)\right] \qquad (5.17)$$

but

$$\left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)+\Phi^{\eta}\left(\frac{-D}{\sigma_1}\right)\right] = 0 \quad \text{if } \eta \text{ is even}$$
$$= 2\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \quad \text{if } \eta \text{ is odd}$$

and

$$\left[\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)+\Phi^{\eta}\left(\frac{-D}{\sigma_2}\right)\right] = 0 \text{ if } \eta \text{ is even}$$
$$= 2\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) \text{ if } \eta \text{ is odd}$$

therefore, note that only the odd terms of the expansion will remain, giving

$$4a^2D^2 \sum_{\substack{\eta=0 \\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!}\, \Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right) \tag{5.18}$$

The next term is

$$a^2D \int_D^{\infty} dv_1 \int_{-D}^{D} v_2\, f(v_1, v_2)\, dv_2 =$$

$$a^2D \sum_{\eta=0}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{ \int_D^{\infty} \Phi^{\eta+1}\left(\frac{v_1}{\sigma_1}\right) dv_1 \right\} \left\{ \int_{-D}^{D} v_2\, \Phi^{\eta+1}\left(\frac{v_2}{\sigma_2}\right) dv_2 \right\} \tag{5.19}$$

which equals

$$a^2D \sum_{\eta=0}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{ -\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \right\} \left\{ D\left[\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)+\Phi^{\eta}\left(\frac{-D}{\sigma_2}\right)\right] - \sigma_2\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)-\Phi^{\eta-1}\left(\frac{-D}{\sigma_2}\right)\right] \right\} \tag{5.20}$$

The reduction of Equation (5.19) into Equation (5.20) is in Appendix III, Section 2. But

$$\left[\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) + \Phi^{\eta}\left(\frac{-D}{\sigma_2}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) \quad \text{if } \eta \text{ is odd}$$

and

$$\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) - \Phi^{\eta-1}\left(\frac{-D}{\sigma_2}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \quad \text{if } \eta \text{ is odd}$$

thus $a^2D \int_D^{\infty} dv_1 \int_{-D}^{D} v_2 \, f(v_1, v_2) \, dv_2 =$

$$2a^2D \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{-\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\right\} \left\{D\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) - \sigma_2 \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right\} \tag{5.21}$$

Now consider the term $- a^2D \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2 \, f(v_1, v_2) dv_2$

This term by similar methods is equal to

$$-2a^2D \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{\Phi^{\eta}\left(\frac{-D}{\sigma_1}\right)\right\} \left\{D\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) - \sigma_2 \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right\} \tag{5.22}$$

But notice that the two Equations, (5.21) and (5.22), can be combined to give

$$-2a^2D \sum_{\eta=0}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{\Phi^{\eta}\left(\frac{-D}{\sigma_1}\right) + \Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\right\} \left\{D\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) - \sigma_2 \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right\} \tag{5.23}$$

but

$$\left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) + \Phi^{\eta}\left(\frac{-D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is odd}$$

Therefore, the two terms, $\alpha^2 D \int_{D}^{\infty} dv_1 \int_{-D}^{D} v_2 f(v_1, v_2) dv_2$ and

$-\alpha^2 D \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2 f(v_1, v_2) dv_2$ are equal to

$$-4\alpha^2 D \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\right\} \left\{D\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) - \sigma_2 \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right\} \tag{5.24}$$

However, by similar means the two terms $-\alpha^2 D \int_{-D}^{\infty} dv_2 \int_{-D}^{D} v_1 f(v_1 v_2) dv_1$

and $-\alpha^2 D \int_{-\infty}^{-D} dv_2 \int_{-D}^{D} v_1 f(v_1, v_2)\, dv_1$ will yield

$$-4\alpha^2 D \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)\right\} \left\{D\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) - \sigma_1 \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\right\} \tag{5.25}$$

Now combine Equations (5.24) and (5.25) to obtain

$$-4\alpha^2 D \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^{\eta}}{\eta!} \left\{ 2D \left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)\right] - \sigma_2 \left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right] - \right.$$

$$\left[\sigma_1 \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right)\right]\Bigg\} \tag{5.26}$$

Now consider the term

$$a^2 \int_{-D}^{D}\int_{-D}^{D} v_1 \quad v_2 \, f(v_1, v_2)\, dv_1 \, dv_2$$

which is equal to

$$a^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \left\{ D\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) + \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right] - \sigma_1\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta-1}\left(\frac{-D}{\sigma_1}\right)\right]\right\}$$

$$\left\{ D\left[\Phi^\eta\left(\frac{D}{\sigma_2}\right) + \Phi^\eta\left(\frac{-D}{\sigma_2}\right)\right] - \sigma_2\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) - \Phi^{\eta-1}\left(\frac{-D}{\sigma_2}\right)\right]\right\} \tag{5.27}$$

by applying Appendix III, Section 2 twice. If the individual components above are observed, it is noted that

$$\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) + \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^\eta\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is odd}$$

$$\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta-1}\left(\frac{-D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is odd}$$

and

$$\left[\Phi^\eta\left(\frac{D}{\sigma_2}\right) + \Phi^\eta\left(\frac{-D}{\sigma_2}\right)\right. = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^\eta\left(\frac{D}{\sigma_2}\right) \qquad \text{if } \eta \text{ is odd}$$

$$\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) - \Phi^{\eta-1}\left(\frac{-D}{\sigma_2}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$

$$= 2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \quad \text{if } \eta \text{ is odd}$$

Then $a^2 \int_{-D}^{D}\int_{-D}^{D} v_1 v_2 \, f(v_1, v_2) \, dv_1 \, dv_2$

$$= 4a^2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!} \left\{D\Phi^\eta\left(\frac{D}{\sigma_1}\right) - \sigma_1\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\right\}\left\{D\Phi^\eta\left(\frac{D}{\sigma_2}\right) - \sigma_2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right\} \quad (5.28)$$

which when multiplied out equals,

$$4a^2D^2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^\eta\left(\frac{D}{\sigma_2}\right) - 4a^2D\sigma_2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)$$

$$+ 4a^2\sigma_1\sigma_2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)$$

$$- 4a^2D\sigma_1 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^\eta\left(\frac{D}{\sigma_2}\right) \quad (5.29)$$

Remembering that $E\left[e_c(t)\right]$ is equal to the sum of the individual terms, this then gives

$$E\left[e_o(t)\right] = 4a^2D^2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^\eta\left(\frac{D}{\sigma_2}\right)$$

$$-8a^2D^2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^\eta\left(\frac{D}{\sigma_2}\right) + 4a^2D\sigma_2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!}\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)$$

$$+4a^2D\sigma_1\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)+4a^2D^2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta}\left(\frac{D}{\sigma_2}\right)$$

$$-4a^2D\sigma_2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)+4a^2\sigma_1\sigma_2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)$$

$$-4a^2D\sigma_1\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta}\left(\frac{D}{\sigma_2}\right) \tag{5.30}$$

It is noticed that some of the terms can be combined, yielding

$$E\left[e_o(t)\right] = 4a^2\sigma_1\sigma_2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^\eta}{\eta!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \tag{5.31}$$

Note that the Equation (5.31) above differs from Equation (4.16), the solution of $E\left[e_o(t)\right]$ in Chapter IV. It will be proven here that these two solutions are identical.

Start with Equation (5.31) and differentiate twice with respect to ρ. Then

$$\frac{\partial^2}{\partial\rho^2}E\left[e_o(t)\right] = 4a^2\sigma_1\sigma^2\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^{\eta-2}}{(\eta-2)!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \tag{5.32}$$

but $\displaystyle\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty}\frac{\rho^{\eta-2}}{(\eta-2)!}\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) =$

$$\frac{1}{4\pi\sqrt{1-\rho^2}}\left\{e^{-\frac{D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}-\frac{2\rho}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]} - e^{-\frac{D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}+\frac{2\rho}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]}\right\} \qquad (5.33)$$

From relationship in Appendix III, Section 4, remembering that

$$\Phi^{\eta+1}(x) \neq \frac{(-1)^\eta}{\sqrt{2\pi}} H_\eta(x)\, e^{\frac{-x^2}{2}}$$

therefore,

$$\frac{\partial^2}{\partial\rho^2} E\left[e_o(t)\right] = \frac{a^2\sigma_1\sigma_2}{\pi\sqrt{1-\rho^2}}\left\{e^{\frac{-D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}-\frac{2\rho}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]} - e^{\frac{-D^2}{2(1-\rho^2)}\left[\frac{1}{\sigma_1^2}+\frac{2\rho}{\sigma_1\sigma_2}+\frac{1}{\sigma_2^2}\right]}\right\} \qquad (5.34)$$

which is equal to Equation (4.12) and, therefore, the two solutions, Equation (5.31) and Equation (4.16), are equivalent.

CHAPTER VI

SIGNAL-TO-NOISE POWER RELATIONSHIP

The signal-to-noise power ratio of a device and especially of an electronic circuit is of interest to an engineer. This chapter will solve for the S/N ratio at the output of the multiplier when the actual clippers are in the circuit. The signal-to-noise power ratio will be:

$$S/N = \frac{\text{signal power}}{(\text{signal} + \text{noise})\text{power} - \text{signal power}} \tag{6.1}$$

$$= \frac{\left\{E\left[e_o(t)\right]\right\}^2}{E\left[\overline{e_o(t)^2}\right] - \left\{E\left[e_o(t)\right]\right\}^2} \tag{6.2}$$

The next step will be to solve for the terms in the above ratio. The results of the solution of the actual clipper, equation (5.31), will give

$$E\left[e_o(t)\right] = 4\alpha^2 \sigma_1\sigma_2 \sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1} \left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1} \left(\frac{D}{\sigma_2}\right)$$

If the above term for the actual clipper is squared, this yields $\left\{E\left[e_o(t)\right]\right\}^2$ which is the signal power term in equation (6.1).

The signal plus noise power term $E\left[\overline{e_o(t)^2}\right]$ can be found in Appendix IV, and is:

$$E\left[\overline{e_o(t)^2}\right] = 8\alpha^4 D^4 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{D}{\sigma_1}\right) \Phi^\eta \left(\frac{D}{\sigma_2}\right) +$$

$$8\alpha^4 D^2 \sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2} \left(\frac{D}{\sigma_2}\right) + 8\alpha^4 D^2 \sigma_1^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2} \left(\frac{D}{\sigma_1}\right) \Phi^\eta \left(\frac{D}{\sigma_2}\right) -$$

$$8a^4D^3\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) - 8a^4D^3\sigma_1 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^\eta \left(\frac{D}{\sigma_2}\right) +$$

$$16a^4D^2\sigma_1\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1} \left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) + 16a^4\sigma_1^2\sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2} \left(\frac{D}{\sigma_1}\right)\Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) -$$

$$16a^4D\sigma_1\sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) - 16a^4D\sigma_1^2\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right)\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \tag{24}$$

These terms can be combined to yield the S/N power ratio. The writer was unsuccessful in reducing the ratio after several attempts. However, in the case of the ideal clippers the S/N power ratio was found in Chapter III to be

$$S/N = \frac{\mu^2}{1 - \mu^2} \qquad \text{where } \mu = \frac{2}{\pi} \sin^{-1} \rho$$

BIBLIOGRAPHY

1. Bierens de Haan, D — NOUVELLES TABLES DE INTEGRALES DEFINIES, Stechert, 1939

2. Cramèr, H. — MATHEMATICAL METHODS OF STATISTICS, Princeton University Press, 1946

3. Dwight, H.B. — TABLES OF INTEGRALS AND OTHER MATHEMATICAL DATA, MacMillan Co., 1934

4. Jahnke, E. and Emde, F. — TABLES OF FUNCTIONS WITH FORMULAE AND CURVES, Dover Publication, 1945

5. Pipes, L. A. — MATRICES IN ENGINEERING, **Electrical Engineering**, September 1937

6. Rice, S. O. — MATHEMATICAL ANALYSIS OF RANDOM NOISE Bell System Technical Journal, Vol. 23, No. 3, pp 282-332, July 1944. Vol. 24, No. 21, pp 46-156, January 1945

7. Robin, L. — FONCTION DE CORRELATION PROPRE ET SPECTRE DE LA DENSITÉ DE PUISSANCE DU BRUIT THERMIQUE ÉCRÉTÉ. FILTRAGE DE SIGNAUX PÉRIODIQUES, SIMPLES, DANS CE BRUIT. Annales des Télécommunications Tome 7, No. 9, Septembre 1952

8. Sheppard, W. F. — ON THE APPLICATION OF THE THEORY OF ERROR TO CASES OF NORMAL DISTRIBUTION AND NORMAL CORRELATION. PTRS, 192(1898) p. 101

9. Stieltjes, T. J. — EXTRAIT D'UNE LETTRE ADRESSÉE À M. HERMITES Bull. Sc. Math., 2:e série, 13(1889) p. 170

10. Terman, F. E. — RADIO ENGINEERING, 3rd edition McGraw-Hill, 1947

PROOF OF ASSUMED EQUATION FOR IDEAL CLIPPER

Figure 11 is a plot of the ideal clipper function. Notice from Figure 11 that the following is true:

$$
\begin{aligned}
e_1(t) &= \alpha D & &\text{when } v_1(t) > 0 \\
&= 0 & &\text{when } v_1(t) = 0 \\
&= -\alpha D & &\text{when } v_1(t) < 0
\end{aligned}
$$

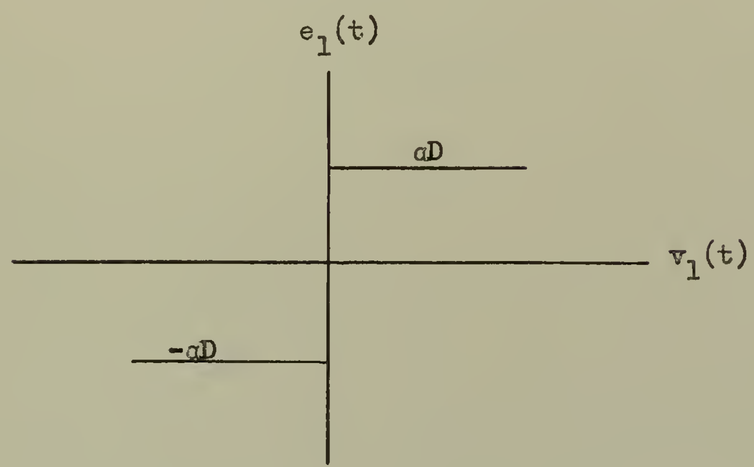

Figure 11. Ideal Clipper's Response

Now assume $e_1(t)$ is

$$e_1(t) = \frac{\alpha D}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega v_1(t)}}{\omega} \, d\omega \tag{1}$$

but $e^{iA} = \cos A + i \sin A$

then

$$e_1(t) = \frac{\alpha D}{\pi i} \int_{-\infty}^{\infty} \frac{\cos v_1(t)\,\omega}{\omega} \, d\omega + \frac{\alpha D}{\pi} \int_{-\infty}^{\infty} \frac{\sin v_1(t)\,\omega}{\omega} \, d\omega \tag{2}$$

but $\displaystyle\int_{-\infty}^{\infty} \frac{\cos v_1(t)\omega}{\omega}\, d\omega = 0$ as cosine is an even function.

Now consider $\displaystyle\int_{-\infty}^{\infty} \frac{\sin v_1(t)\omega}{\omega}\, d\omega_1$

$$\int_{-\infty}^{\infty} \frac{\sin v_1(t)\omega}{\omega}\, d\omega = \pi \text{ when } v_1(t) > 0 \tag{3}$$

$$= 0 \text{ when } v_1(t) = 0$$

$$= -\pi \text{ when } v_1(t) < 0$$

Then

$$\begin{aligned} e_1(t) &= aD && \text{when } v_1(t) > 0 \\ &= 0 && \text{when } v_1(t) = 0 \\ &= -aD && \text{when } v_1(t) < 0 \end{aligned}$$

Thus it is seen that the assumed Equation (1) represents the relationship between the input and output of the ideal clipper.

APPENDIX II

PROOF OF ASSUMED EQUATION FOR ACTUAL CLIPPER

Note from Figure 12 that the slope of $e_1(t)$ is equal to a between points -D and D and outside of these two points the slope is zero.

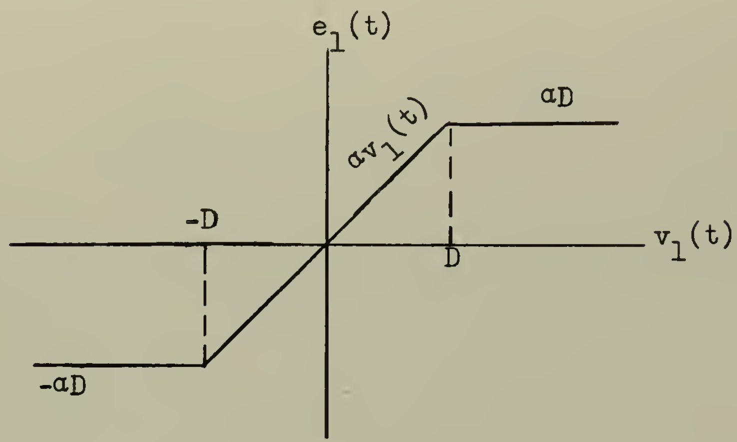

Figure 12. Actual Clipper's Response

Now assume $e_1(t)$ is,

$$e_1(t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\sin v_1(t)u \sin Du}{u^2} \, du \tag{4}$$

Take the derivative of the above equation (4), the assumed function of $e_1(t)$, with respect to $v_1(t)$, and this gives

$$\frac{de_1(t)}{dv_1(t)} = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos v_1(t)u \sin Du}{u} \, du \tag{5}$$

but the right-hand side of equation (5) is an even function, therefore,

$$\frac{de_1(t)}{dv_1(t)} = \frac{2a}{\pi} \int_{\cancel{-\infty}\,0}^{\infty} \frac{\cos v_1(t)u \sin Du}{u} \, du \tag{6}$$

which is equal to

$$= 0 \quad \text{if } v_1(t) > D$$

$$= \frac{2a}{\pi}\left[\frac{\pi}{2}\right] = a \quad \text{if } v_1(t) < D$$

Thus it is noted that the assumed equation

$$e_1(t) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\sin v_1(t)u \sin Du}{u^2}\, du$$

gives the following slopes:

a when $-D < v_1(t) < D$

0 when $-D > v_1(t) > D$

Hence, the assumed equation (4) represents the relationship between the input and output of the actual clipper.

APPENDIX III

MATHEMATICAL RELATIONSHIPS

A few of the mathematical relationships used in the different chapters are assembled here in order to simplify the mathematics of the individual chapters. Section 1 is from basic definitions. Section 2 and Section 3 are derived by partial integrations. Section 4 is an extension of one of Cramèr's formulas.

Section 1.

$$\Phi^{\eta}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^{\eta}}{dx^{\eta}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \tag{7}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{d^{\eta}}{dx^{\eta}} (e^{-x^2/2})$$

$$= \frac{(-1)^{\eta-1}}{\sqrt{2\pi}} H_{\eta-1}(x)\, e^{-x^2/2} \tag{8}$$

Section 2.

$$\int_{-D}^{D} \frac{v_1}{\sigma_1} \Phi^{\eta+1} (\frac{v_1}{\sigma_1}) dv_1 = \sigma_1 \int_{-D/\sigma_1}^{D/\sigma_1} x\Phi^{\eta+1} (x)\, dx \tag{9}$$

$$= \sigma_1 \left\{ x\Phi^{\eta}(x) \Big|_{-D/\sigma_1}^{D/\sigma_1} - \int_{-D/\sigma_1}^{D/\sigma_1} \Phi^{\eta}(x)\, dx \right\}$$

$$= \sigma_1 \left\{ \frac{D}{\sigma_1} \left[\Phi^{\eta}(\frac{D}{\sigma_1}) + \Phi^{\eta}(-\frac{D}{\sigma_1})\right] - \left[\Phi^{\eta-1}(\frac{D}{\sigma_1}) - \Phi^{\eta-1}(-\frac{D}{\sigma_1})\right] \right\}$$

But

$$\left[\Phi^{\eta} (\frac{D}{\sigma_1}) + \Phi^{\eta}(-\frac{D}{\sigma_1})\right] = 2\Phi^{\eta} (\frac{D}{\sigma_1}) \quad \text{if } \eta \text{ is odd}$$

$$= 0 \quad \text{if } \eta \text{ is even}$$

and

$$\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta-1}\left(-\frac{D}{\sigma_1}\right)\right] = 2\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is odd}$$

$$= 0 \qquad \text{if } \eta \text{ is even}$$

therefore,

$$\int_{-D}^{D} \frac{v_1}{\sigma_1} \Phi^{\eta+1}\left(\frac{v_1}{\sigma_1}\right) dv_1 = 2\left[D\,\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) - \sigma_1\,\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right)\right] \qquad \text{if } \eta \text{ is odd}$$

$$= 0 \qquad \text{if } \eta \text{ is } \cancel{\text{odd}}\ \text{even} \tag{10}$$

Section 3.

$$\sigma_1^2 \int_{-D}^{D} \frac{v_1^2}{\sigma_1^2} \Phi^{\eta+1}\left(\frac{v_1}{\sigma_1}\right) \frac{dv_1}{\sigma_1} = \sigma_1^2 \int_{-D/\sigma_1}^{D/\sigma_1} x^2\, \Phi^{\eta+1}(x)\, dx \tag{11}$$

$$= \sigma_1^2\, x^2\, \Phi^{\eta}(x) \Big|_{-D/\sigma_1}^{D/\sigma_1} - 2\sigma_1^2 \int_{-D/\sigma_1}^{D/\sigma_1} x\, \Phi^{\eta}(x)\, dx$$

$$= \sigma_1^2\, x^2\, \Phi^{\eta}(x) \Big|_{-D/\sigma_1}^{D/\sigma_1} - 2\sigma_1\, x\, \Phi^{\eta-1}(x) \Big|_{-D/\sigma_1}^{D/\sigma_1} + 2\sigma_1^2 \int_{-D/\sigma_1}^{D/\sigma_1} \Phi^{\eta-1}(x)dx$$

$$= D^2\left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta}\left(-\frac{D}{\sigma_1}\right)\right] - 2D\sigma_1\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) + \Phi^{\eta-1}\left(-\frac{D}{\sigma_1}\right)\right]$$

$$+ 2\sigma_1^2\left[\Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta-2}\left(-\frac{D}{\sigma_1}\right)\right] \tag{12}$$

but notice that

$$\left[\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta}\left(-\frac{D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is odd}$$

$$= 2\Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is even}$$

$$\left[\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) + \Phi^{\eta-1}\left(-\frac{D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is odd}$$

$$= 2\Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is even}$$

$$\left[\Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) - \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is odd}$$

$$= 2\Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is even}$$

Therefore, the equation will have even terms only and can be written

$$\int_{-D}^{D} v_1^2\, \Phi^{\eta+1}\left(\frac{v_1}{\sigma_1}\right) dv_1 =$$

$$2D^2\, \Phi^{\eta}\left(\frac{D}{\sigma_1}\right) - 4D\sigma_1\, \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) + 4\sigma_1^2\, \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right)$$

where η is even.

Section 4.

$$\sum_{\eta=0}^{\infty} \frac{H_\eta(x)\, H_\eta(y)}{\eta!}\, t^\eta = \frac{1}{\sqrt{1 - t^2}}\, e^{-\frac{t^2x^2+t^2y^2-2txy}{2(1-t^2)}} \tag{14}$$

$$\text{if } |t| < 1$$

Equation (14) is a restatement of one of Cramèr's equations. Note that, if t is replaced by $-t$, all the even terms remain the same while the odd terms are negatively changed. Therefore, if the above equation is subtracted from the original equation, with t replaced by $-t$, this will give twice the odd terms only.

Then,

$$\sum_{\substack{\eta=0\\ \text{odd}}}^{\infty} \frac{H_\eta(x)\, H_\eta(y)}{\eta!}\, t^\eta = \frac{1}{2\sqrt{1-t^2}}\, e^{-\frac{t^2x^2+t^2y^2}{2(1-t^2)}} \left[e^{\frac{txy}{1-t^2}} - e^{-\frac{txy}{1-t^2}}\right] \tag{15}$$

$$= \frac{1}{\sqrt{1 - t^2}}\, e^{-\frac{t^2x^2+t^2y^2}{2(1-t^2)}} \sinh \frac{txy}{1 - t^2} \tag{16}$$

APPENDIX IV

SECOND MOMENT OF $e_o(t)$

The application of Cramèr's method, similar to the method employed in Chapter V, will be used to solve for the second moment of $e_o(t)$ for the actual clippers. The sub-regions of Figure 10 will be considered and the equation for the second moment of each of these sub-regions will be written. Then $E\left[\overline{e_o(t)}^2\right]$, the second moment of $e_o(t)$, will be the sum of the contributions of each sub-region.

The equations for the sub-regions follow:

Area 1:

$$E\left[\overline{e_{o_1}(t)}^2\right] = \alpha^4 D^4 \int_D^\infty \int_D^\infty f(v_1, v_2)\, dv_1\, dv_2$$

Area 2:

$$E\left[\overline{e_{o_2}(t)}^2\right] = \alpha^4 D^2 \int_D^\infty dv_1 \int_{-D}^{D} v_2^2 f(v_1, v_2)\, dv_2$$

Area 3:

$$E\left[\overline{e_{o_3}(t)}^2\right] = \alpha^4 D^4 \int_{-\infty}^{-D} dv_2 \int_D^\infty f(v_1, v_2)\, dv_1$$

Area 4:

$$E\left[\overline{e_{o_4}(t)}^2\right] = \alpha^4 D^2 \int_{-D}^{D} dv_1 \int_D^\infty v_1^2 f(v_1, v_2)\, dv_2$$

Area 5:

$$E\left[\overline{e_{o_5}(t)}^2\right] = \alpha^4 \int_{-D}^{D} \int_{-D}^{D} v_1^2\, v_2^2 f(v_1, v_2)\, dv_1\, dv_2$$

Area 6:

$$E\left[\overline{e_{o_6}(t)}^2\right] = a^4D^2 \int_{-D}^{D} dv_1 \int_{-\infty}^{-D} v_1^2\, f(v_1, v_2)\, dv_2$$

Area 7:

$$E\left[\overline{e_{o_7}(t)}^2\right] = a^4D^4 \int_{-\infty}^{-D} dv_1 \int_{D}^{\infty} f(v_1, v_2)\, dv_2$$

Area 8:

$$E\left[\overline{e_{o_8}(t)}^2\right] = a^4D^2 \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2^2\, f(v_1, v_2)\, dv_2$$

Area 9:

$$E\left[\overline{e_{o_9}(t)}^2\right] = a^4D^4 \int_{-\infty}^{-D} \int_{-\infty}^{-D} f(v_1, v_2)\, dv_1\, dv_2$$

but

$$E\left[\overline{e_o(t)}^2\right] = E\left[\overline{e_{o_1}(t)}^2\right] + E\left[\overline{e_{o_2}(t)}^2\right] + E\left[\overline{e_{o_3}(t)}^2\right] + E\left[\overline{e_{o_4}(t)}^2\right] + E\left[\overline{e_{o_5}(t)}^2\right]$$
$$+ E\left[\overline{e_{o_6}(t)}^2\right] + E\left[\overline{e_{o_7}(t)}^2\right] + E\left[\overline{e_{o_8}(t)}^2\right] + E\left[\overline{e_{o_9}(t)}^2\right]$$

then

$$E\left[\overline{e_o(t)}^2\right] = a^4D^4 \int_{D}^{\infty} \int_{D}^{\infty} f(v_1, v_2)dv_1\, dv_2 + a^4D^2 \int_{D}^{\infty} dv_1 \int_{-D}^{D} v_2^2\, f(v_1, v_2)dv_2$$

$$+ a^4D^4 \int_{-\infty}^{-D} dv_2 \int_{D}^{\infty} f(v_1, v_2)dv_1 + a^4D^2 \int_{-D}^{D} dv_1 \int_{D}^{\infty} v_1^2\, f(v_1, v_2)dv_2 +$$

$$a^4 \int_{-D}^{D} \int_{-D}^{D} v_1^2 \, v_2^2 \, f(v_1, v_2) dv_1 \, dv_2 + a^4 D^2 \int_{-D}^{D} dv_1 \int_{-\infty}^{-D} v_1^2 f(v_1, v_2) dv_2$$

$$+ a^4 D^2 \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2^2 f(v_1, v_2) dv_2 + a^4 D^4 \int_{-\infty}^{-D} dv_1 \int_{D}^{\infty} f(v_1, v_2) dv_2$$

$$+ a^4 D^4 \int_{-\infty}^{-D} \int_{-\infty}^{-D} f(v_1, v_2) dv_1 \, dv_2 \tag{17}$$

Now consider the above equation term by term. The results of this term by term analysis will later be combined to obtain $E\left[\overline{e_o(t)}^2\right]$.

$$a^4 D^4 \int_{D}^{\infty} \int_{D}^{\infty} f(v_1, v_2) dv_1 \, dv_2 = a^4 D^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{D}{\sigma_1}\right) \Phi^\eta \left(\frac{D}{\sigma_2}\right)$$

$$a^4 D^4 \int_{-\infty}^{-D} \int_{-\infty}^{-D} f(v_1, v_2) dv_1 \, dv_2 = a^4 D^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{-D}{\sigma_1}\right) \Phi^\eta \left(\frac{-D}{\sigma_2}\right)$$

$$a^4 D^4 \int_{D}^{\infty} dv_1 \int_{-\infty}^{-D} f(v_1, v_2) dv_2 = -a^4 D^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{D}{\sigma_1}\right) \Phi^\eta \left(\frac{-D}{\sigma_2}\right)$$

$$a^4 D^4 \int_{-\infty}^{-D} dv_1 \int_{D}^{\infty} f(v_1, v_2) dv_2 = -a^4 D^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta \left(\frac{-D}{\sigma_1}\right) \Phi^\eta \left(\frac{D}{\sigma_2}\right)$$

but notice that the above functions of Φ can be combined to give

$$\alpha^4 D^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!}\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) - \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right]\left[\Phi^\eta\left(\frac{D}{\sigma_2}\right) - \Phi^\eta\left(\frac{-D}{\sigma_2}\right)\right]$$

but $\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) - \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right] = 2\Phi^\eta\left(\frac{D}{\sigma_1}\right)$ if η is even

$= 0$ if η is odd

and

$\left[\Phi^\eta\left(\frac{D}{\sigma_2}\right) - \Phi^\eta\left(\frac{-D}{\sigma_2}\right)\right] = 2\Phi^\eta\left(\frac{D}{\sigma_2}\right)$ if η is even

$= 0$ if η is odd

Thus observe that only even terms will result in the solution and hence the four terms combine to give

$$4\alpha^4 D^4 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!}\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right)\Phi^\eta\left(\frac{D}{\sigma_2}\right)\right] \tag{18}$$

Now take next the term

$$\alpha^4 D^2 \int_D^{\infty} dv_1 \int_{-D}^{D} v_2^2\, f(v_1, v_2) dv_2$$

which is equal to

$$\alpha^4 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!}\left[-\Phi^\eta\left(\frac{D}{\sigma_1}\right)\right]\left[2D^2\Phi^\eta\left(\frac{D}{\sigma_2}\right) - 4D\sigma_2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) + 4\sigma_2^2\, \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right)\right] \tag{19}$$

From application of Appendix III, Section 1 and Section 3; consider the term

$$\alpha^4 D^2 \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2^2\, f(v_1, v_2)\, dv_2$$

which equals, also from Appendix III, Section 1 and Section 3,

$$\alpha^4 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!}\left[-\Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right]\left[2D^2\Phi^\eta\left(\frac{D}{\sigma_2}\right) - 4D\sigma_2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right.$$
$$\left. + 4\sigma_2^2\Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right)\right] \tag{20}$$

But Equations (19) and (20) can be combined to give

$$-\alpha^4 D^2 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!}\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) + \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right]\left[2D^2\Phi^\eta\left(\frac{D}{\sigma_2}\right) - 4D\sigma_2\Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right)\right.$$
$$\left. + 4\sigma_2^2\Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right)\right] \tag{21}$$

but

$$\left[\Phi^\eta\left(\frac{D}{\sigma_1}\right) + \Phi^\eta\left(\frac{-D}{\sigma_1}\right)\right] = 0 \qquad \text{if } \eta \text{ is even}$$
$$= 2\Phi^\eta\left(\frac{D}{\sigma_1}\right) \qquad \text{if } \eta \text{ is odd}$$

which will give a contribution that is a product of an even term multiplied by an odd term. However, it is known that an even term multiplied by an odd term will not yield a contribution to the solution of $E\left[\overline{e_o(t)}^2\right]$.

Hence these two terms, $\alpha^4 D^2 \int_D^\infty dv_1 \int_{-D}^{D} v_2^2 \; f(v_1, v_2)dv_2$ and

$\alpha^4 D^2 \int_{-\infty}^{-D} dv_1 \int_{-D}^{D} v_2^2 \; f(v_1, v_2)dv_2$ do not contribute to the solution of the second moment of $e_o(t)$.

By similar reasoning, the two terms

$$\alpha^4 D^2 \int_{D}^{\infty} dv_2 \int_{-D}^{D} v_1^2 \, f(v_1, v_2) dv_1 \text{ and } \alpha^4 D^2 \int_{-\infty}^{-D} dv_2 \int_{-D}^{D} v_1^2 \, f(v_1, v_2) dv_1$$

are observed to yield no contribution to the solution of $E\left[e_o(t)\right]^2$.

Next consider the term $\alpha^4 \int_{-D}^{D}\int_{-D}^{D} v_1^2 v_2^2 \, f(v_1, v_2) dv_1 dv_2$.

By applying Appendix III, Section 3 twice, it is equal to

$$\alpha^4 \sum_{\eta=0}^{\infty} \frac{\rho^\eta}{\eta!} \left[2D^2 \Phi^\eta\left(\frac{D}{\sigma_1}\right) - 4D\sigma_1 \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) + 4\sigma_1^2 \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right)\right]$$

$$\left[2D^2 \Phi^\eta\left(\frac{D}{\sigma_2}\right) - 4D\sigma_2 \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) + 4\sigma_2^2 \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right)\right] \qquad (22)$$

But it is known that each multiplicand can be even only. Then Equation (22) by expansion is equal to

$$4\alpha^4 D^4 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{D}{\sigma_1}\right) \Phi^\eta\left(\frac{D}{\sigma_2}\right) - 8\alpha^4 D^3 \sigma_2 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \Phi^\eta\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) \frac{\rho^\eta}{\eta!} +$$

$$8\alpha^4 D^2 \sigma_2^2 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^\eta\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) - 8\alpha^4 D^3 \sigma_1 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^\eta\left(\frac{D}{\sigma_2}\right) +$$

$$16\alpha^4 D^2 \sigma_1 \sigma_2 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) - 16\alpha^4 D \sigma_1^2 \sigma_2 \sum_{\substack{\eta=0 \\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) +$$

$$8a^4D^2\sigma_1^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right) - 16a^4D\sigma_1^2\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) +$$

$$16a^4\sigma_2^2\sigma_1^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) \tag{23}$$

Now remembering that the second moment of $e_o(t)$, $E\left[\overline{e_o(t)}^2\right]$, is equal to the sum of the contributions from each sub-region, one gets

$$E\left[\overline{e_o(t)}^2\right] = 8a^4D^4 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right) -$$

$$8a^4D^3\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) + 8a^4D^2\sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) -$$

$$8a^4D^3\sigma_1 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right) + 16a^4D^2\sigma_1\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) -$$

$$16a^4D\sigma_1\sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-1}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) + 8a^4D^2\sigma_1^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta}\left(\frac{D}{\sigma_2}\right) -$$

$$16a^4D\sigma_1^2\sigma_2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-1}\left(\frac{D}{\sigma_2}\right) + 16a^4\sigma_1^2\sigma_2^2 \sum_{\substack{\eta=0\\ \text{even}}}^{\infty} \frac{\rho^\eta}{\eta!} \Phi^{\eta-2}\left(\frac{D}{\sigma_1}\right) \Phi^{\eta-2}\left(\frac{D}{\sigma_2}\right) \tag{24}$$

APPENDIX V

EVALUATION OF "$\frac{\partial}{\partial x} \psi(x) \Big|_{x=0}$"

In this case it will be necessary to start with equation (4.6).

Then

$$\psi(x) = -\frac{2\alpha^2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin D\omega_1 \sin D\omega_2}{\omega_1^2\ \omega_2^2}\Big\{e^{-\frac{1}{2}(\sigma_1^2\omega_1^2+2\sigma_1\sigma_2\omega_1\omega_2 x+\sigma_2^2\omega_2^2)} - e^{-\frac{1}{2}(\sigma_1^2\omega_1^2 - 2\sigma_1\sigma_2\omega_1\omega_2 x + \sigma_2^2\omega_2^2)}\Big\}\, d\omega_1\, d\omega_2 \tag{25}$$

then the partial derivative yields

$$\frac{\partial}{\partial x}\psi(x) = +\frac{2\alpha^2\sigma_1\sigma_2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin D\omega_1 \sin D\omega_2}{\omega_1\omega_2}\Big\{e^{-\frac{1}{2}(\sigma_1^2\omega_1^2+2\sigma_1\sigma_2\omega_1\omega_2 x+\sigma_2^2\omega_2^2)} + e^{-\frac{1}{2}(\sigma_1^2\omega_1^2 - 2\sigma_1\sigma_2\omega_1\omega_2 x + \sigma_2^2\omega_2^2)}\Big\}\, d\omega_1\, d\omega_2 \tag{26}$$

Then to evaluate for $x = 0$, one gets

$$\frac{\partial}{\partial x}\psi(x)\Big|_{x=0} = \frac{4\alpha^2\sigma_1\sigma_2}{\pi^2}\Big\{\int_{-\infty}^{\infty}\frac{\sin D\omega_1}{\omega_1}\, e^{-\frac{1}{2}\sigma_1^2\omega_1^2}\, d\omega_1\Big\} \times \Big\{\int_{-\infty}^{\infty}\frac{\sin D\omega_2}{\omega_2}\, e^{-\frac{1}{2}\sigma_2^2\omega_2^2}\, d\omega_2\Big\} \tag{27}$$

But from Cramèr [2] one obtains the general term

$$\int_{-\infty}^{\infty} e^{itx-\frac{1}{2}hx^2}\, dx = \sum_{0}^{\infty}\frac{(it)^\eta}{\eta!}\int_{-\infty}^{\infty} x^\eta\, e^{-\frac{1}{2}hx^2}\, dx = \sqrt{\frac{2\pi}{h}}\, e^{-\frac{t^2}{2h}} \tag{28}$$

Then, applying this to the present case, it yields for one term

$$\int_{-\infty}^{\infty} \cos tx \, e^{-\frac{x^2\sigma_1^2}{2}} \, dx = \frac{\sqrt{2\pi}}{\sigma_1} \, e^{-\frac{t^2}{2\sigma_1^2}} \tag{29}$$

but if both sides are integrated with respect to t between the limits of 0 and D, one obtains

$$\int_0^D dt \int_{-\infty}^{\infty} \cos tx \, e^{-\frac{x^2\sigma_1^2}{2}} \, dx = \frac{\sqrt{2\pi}}{\sigma_1} \int_0^D e^{-\frac{t^2}{2\sigma_1^2}} \, dt \tag{30}$$

But the left-hand side of the above equation equals (by inversion of the order of integration of the repeated integral)

$$\int_{-\infty}^{\infty} e^{-\frac{x^2\sigma_1^2}{2}} \, dx \int_0^D \cos tx \, dt$$

but

$$\int_0^D \cos tx \, dt = \frac{1}{x} \sin tx \Big|_0^D = \frac{\sin Dx}{x} \tag{31}$$

then

$$\int_{-\infty}^{\infty} \frac{\sin Dx}{x} \, e^{-\frac{x^2\sigma_1^2}{2}} \, dx = \frac{\sqrt{2\pi}}{\sigma_1} \int_0^D e^{-\frac{t^2}{2\sigma_1^2}} \, dt \tag{32}$$

By applying this to the case under consideration, it results in

$$\int_{-\infty}^{\infty} \frac{\sin D\omega_1 \sin D\omega_2}{\omega_1\omega_2} \, e^{-\frac{1}{2}(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2)} \, d\omega_1 \, d\omega_2 =$$

$$\frac{2\pi}{\sigma_1\sigma_2} \left\{ \int_0^D e^{-\frac{t^2}{2\sigma_1^2}} \, dt \right\} \left\{ \int_0^D e^{-\frac{t^2}{2\sigma_2^2}} \, dt \right\} \tag{33}$$

$$= 4\pi \left\{ \operatorname{erf} \frac{D}{\sigma_1\sqrt{2}} \right\} \left\{ \operatorname{erf} \frac{D}{\sigma_2\sqrt{2}} \right\} \tag{34}$$

where

$$\operatorname{erf} \frac{D}{\sigma_1 \sqrt{2}} = \int_0^{D/\sigma_1 \sqrt{2}} e^{-x^2}\, dx$$

and

$$\operatorname{erf} \frac{D}{\sigma_2 \sqrt{2}} = \int_0^{D/\sigma_2 \sqrt{2}} e^{-x^2}\, dx$$